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Error Probabilities for Rician Fading Multichannel Reception of Binary and N-ary Signals

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ABSTRACT

Performance characteristics are derived for two different forms of multireceivers (the coherent and noncoherent) which are used with binary and N-ary signaling through the Rician fading multichannel. In this model it is presumed that each transmission mode supports a specular component of distinct strength and a random or scatter component which fades according to the Rayleigh distribution. Heretofore, performance analyses of multichannel links have assumed that the fading obeys the Rayleigh law. This multichannel model is sufficiently general to include four types of practical multichannels: the Rician, Rayleigh, fixed- and/or mixed-mode multichannels.

Error probabilities are graphically illustrated and compared for various multichannel models. These results show that multichannel reception increases the reliability of communication as compared with single-channel reception. It is found that the effectiveness of multichannel reception is highly dependent on the strength of the specular channel component and the mean squared value of the random channel component. In particular, multichannel reception is more effective when applied to the completely random multichannel.

Finally, asymptotic expansions for system performance are derived for various multichannel conditions. These results indicate the rapidity with which system performance increases or decreases as the multichannel characteristics change. For special cases the error-rate expressions, as well as the asymptotic expressions, reduce to well-known results.

I. INTRODUCTION

The performance of N-ary multichannel communication systems operating in noisy-multiplicative environments is of paramount importance to the system design engineer. Considered here are the multireceiver performance characteristics for a binary coherent multichannel communication system using either orthogonal equal-energy equiprobable signals or correlated equal-energy equiprobable signaling techniques and the N-ary performance characteristics for a noncoherent multichannel communication system using orthogonal signals. The transmitters, for either of the multireceiver terminations, operate through the Rician fading multichannel. In multichannel communications digital information is transmitted over a number of channels in lieu of relying on a simple propagation mode between the transmitter and receiver.[1] Examples of this type of communication are communications via the ionosphere and/or troposphere both above and below the maximum usable frequency, lunar relay links, and deep-space probe telemetry systems. In this type of transmission the propagation modes experience severe fades from time to time (Ref. 2 and 3). The usual procedure employed with this type of radio link is that of diversity reception.[2] The multichannel model presumed also depicts a resolvable multipath situation (Ref. 4).

[1]The term multichannel is borrowed from a paper by Price (Ref. 1). It is used here to include diversity reception and resolvable multipath reception. Another application, pointed out by Price, is in situations where it is impractical to employ a single transmitter-antenna combination or a phased array of antennas and transmitters to handle the transmitter power capabilities.

[2]Four distinct types of diversity reception are included here: diversity as to time, as to frequency, as to space, and polarization diversity.

The multichannel model considered is sufficiently general to include four types of practical multichannels. These are the Rician fading multichannel supporting equally reliable propagation modes, the fixed-mode multichannel, the Rician mixed-mode multichannel, and the Rayleigh fading or completely random multichannel. Heretofore, when fading conditions have been assumed to exist in the multichannel, the fading phenomena have been considered, except in special cases, to obey the Rayleigh distribution (Ref. 5–8).

The purpose of this Report is to present system performance characteristics based on evaluation of the system error probability. In fact, the principal results of this study are rather general expressions for this error probability. These expressions differ from previous work in that they include a wider class of fading phenomena. In certain special cases, as will be pointed out, results reduce to previously derived results.

Another goal of this Report is to present asymptotic performance characteristics for various multichannel models. By "asymptotic performance characteristics" is meant system error-rate behavior at low probabilities of committing an error.

Finally, numerical computations are presented for various multichannel conditions. These computations provide the system designer with information concerning the effects of fixed and random multichannel components on over-all system performance. These characteristics are of interest in determining the required transmitter output power for certain a priori known tolerable error rates and channel conditions.

II. DETAILED DESCRIPTION OF THE COMMUNICATION LINK

In this discussion of the various parts of the system, a brief description is given of the signals selected for transmission by each transmitter and of the multichannel model, followed by a more detailed discussion of the multireceivers.

A. *The Signals*

Stored at the ith transmitter is the set $\{s_{ki}(t)\}$, $k=1, 2, \cdots N$ of signals used for conveying information to the multireceiver. We assume that all signals are limited in time to some interval, say $0 \leq t \leq T$. Within this interval, however, the wave shapes are arbitrary. We further assume that the signal set has equal cross-correlation coefficients, i.e.,

$$\lambda = \frac{1}{2E_i}\left| \int_0^T s_{ki}(t) \quad s_{ji}(t)dt \right| \tag{1}$$

where E_i is the signal energy transmitted by the ith transmitter and may be defined by[3]

$$E_i = \frac{1}{2}\int_0^T \left| s_{ki}^2(t) \right| dt \tag{2}$$

for $i = 1,2,\cdots M$. Finally, we restrict the signals to occur with equal probability, and in the noncoherent multireceiver termination, the quantity λ is assumed to be zero; i.e., the transmitted waveforms are orthogonal. The multireceiver is presumed to have at its disposal replicas of the N signaling waveforms and a statistical knowledge of the transmitted signals, i.e., a priori probabilities of occurrence.

B. *The Multichannel Model*[4]

The multilink channel to which the transmitter and multireceiver are presumed to be connected is shown in Fig. 1. The model for the ith propagation mode may be best understood by indicating what happens to a signal which passes through it. In accordance with the signaling structure stored at the ith transmitter the output of the ith channel is assumed to be represented by[5]

$$x_{ki}(t, \theta_i, a_i, \tau_i) = a_i \, s_{ki}(t-\tau_i) \exp\left[j(\omega_0 t + \theta_i)\right] \tag{3}$$

[3]See Section III, Conclusions, regarding the assumption of equal energies among signals as well as the energy allotted to each channel.

[4]The multichannel model briefly described here was postulated by G. Turin. For a more complete and thorough description see Ref. 4 and 9.

[5]We have assumed that the ith channel induces no doppler shift on the transmitted waveform. If it should, however, we presume that the multireceiver knows the exact amount of this shift.

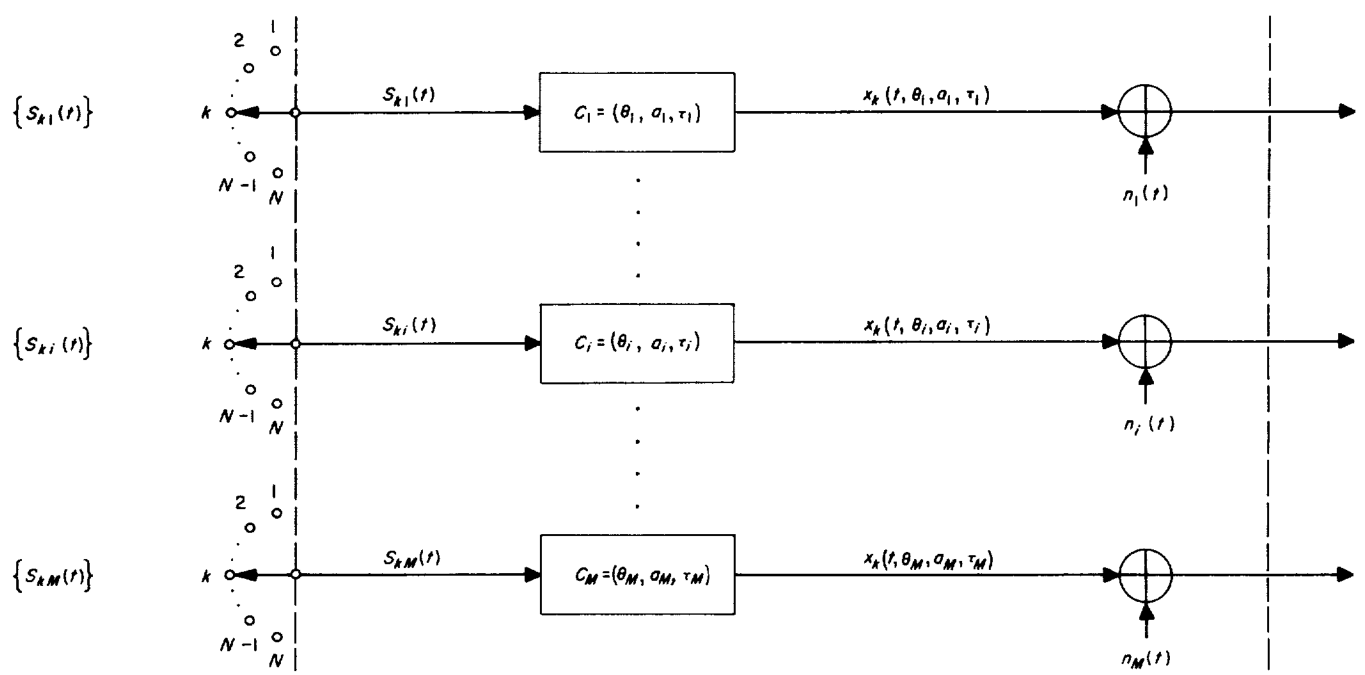


Fig. 1. A multilink channel

where we have assumed that $s_{ki}(t)$ was transmitted and $i = 1,2,\cdots M$. That is, the fading medium is characterized by three statistically independent vector quantities: the multichannel gain $\mathbf{a} = (a_1, a_2, \cdots a_M)$; the multichannel phase characteristic $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots \theta_M)$; and the modulation delay characteristic $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots \tau_M)$. These vectors are assumed to be random and consequently must be described in terms of probability density functions. Furthermore, we assume that the fading is slow in comparison with the signal interval of T sec, and that the propagation modes are mutually independent and statistically independent of the additive noise $n_i(t)$. Consequently, the first-order statistic $p(a_i, \theta_i)$ is sufficient to describe the fading medium. In either multireceiver termination we assume that $\boldsymbol{\tau}$ is known exactly, whereas the multichannel gain and the multichannel phase characteristic, if known to the multireceiver, are obtained as a result of measurement. The output of the ith channel may be broken up into two components: a fixed or specular component and a completely random or scatter component, i.e.,

$$x_{ki}(t) = s_{ki}(t - \tau_i)\,[\underbrace{\alpha_i e^{-j\delta_i}}_{\text{specular component}} + \underbrace{S\, e^{-j\phi}}_{\text{random or scatter component}}]\, e^{j\omega_0 t} \qquad (4)$$

where S and ϕ are independent random variables, the first obeying the Rayleigh density function with mean square $2\sigma^2$, and the second obeying the uniform density function with uniformly distributed phase over an interval of length 2π. It is easily shown (Ref. 9 and 10) that the joint distribution of the length and angle of the sum of the fixed vector (α_i, δ_i) and the random vector (S, ϕ) is

$$p(a_i,\theta_i) = \frac{a_i}{2\pi\sigma^2}\exp\left[-\frac{a_i^2 + \alpha_i^2 - 2\alpha_i a_i \cos(\theta_i - \delta_i)}{2\sigma^2}\right] \qquad 0 \le a_i \le \infty$$

$$= 0 \quad \text{elsewhere} \qquad 0 < (\theta_i - \delta_i) \le 2\pi \qquad (5)$$

The three channel parameters α_i, σ, and δ_i of Eq. (5) may be given physical interpretations. The quantity α_i may be considered to be the strength of the fixed (specular) component in the ith channel, and $2\sigma^2$ is the mean squared value of the random or scatter component in the ith channel. For convenience we define $\gamma_i^2 = \alpha_i^2/2\sigma^2$ as the ratio of the average energy received via the fixed-channel component to the average energy received via the random component. Thus the multichannel model is sufficiently general to represent several types of diversity operation, e.g., frequency, time, antenna, or the resolvable multipath situation (Ref. 4). The distribution (Eq. 5) is sufficiently general, since as the parameter γ_i^2 approaches zero (absence of specular channel components) we have the Rayleigh distribution, while if γ_i^2 approaches infinity (presence of specular components) it may be first approximated by the Gaussian density function of mean α_i and variance $2\sigma^2$, and in the limit by the delta function $\delta(a_i - \alpha_i)$.

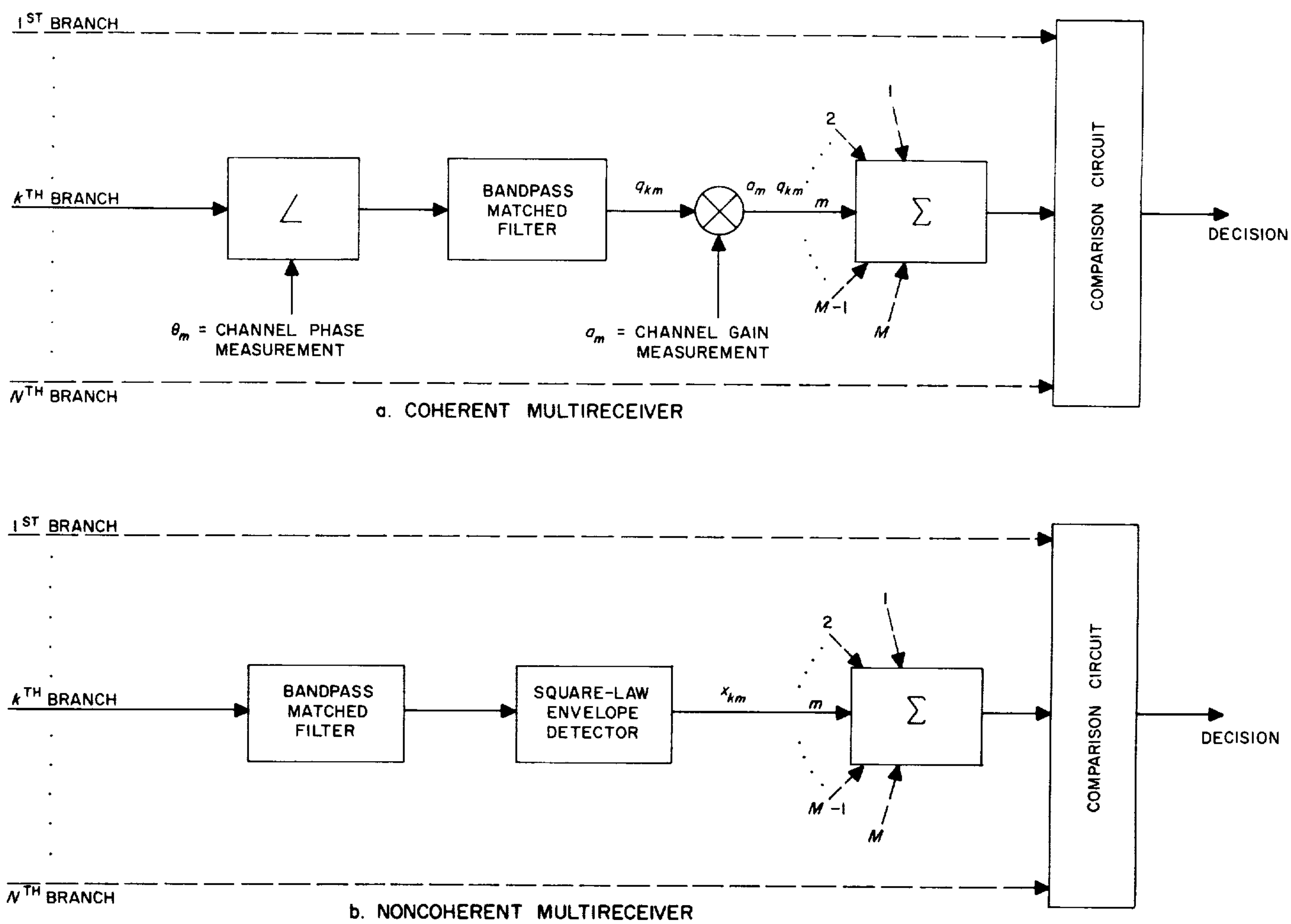
Experimental justification indicating the validity of Eq. (5) is given fully in the propagation literature (see Ref. 2 and 3). The expression typifies channel conditions for both ionospheric and tropospheric radio links operating above and below the maximum usable frequency. It may be used, as a good approximation, for representing the received signal strength and carrier phase shift of a signal received from a tumbling satellite or possibly orbital chaff. Another interesting example of this type of channel would be communications via the lunar surface. Radar returns from the lunar surface indicate that most of the power reflected from the Moon is returned by specular reflection even though some of the signal power is returned from the lunar surface out to the limb (Ref. 11 and 12). The limb reflections account for the fading and correspond to a large γ_i^2 in Eq. (4). Hence, such a model depicts communications via the lunar surface.

After passing through the fading medium, the transmitted signal is further perturbed by additive white, stationary, Gaussian noise having a flat power density of N_0 w/cps single sided. The additive noise is assumed to be statistically independent from channel to channel and has the same rms value in each channel. Hence at the output of the ith channel, presuming $s_{ki}(t)$ is transmitted, we have $y_i(t) = x_{ki}(t) + n_i(t)$.

C. *The Multireceivers*

The multireceivers, which are connected to the multichannel just described, are shown in Fig. 2a and 2b. In the coherent termination it is assumed that the multireceiver has available the necessary equipment for measuring the multichannel gain and phase characteristic. Under such conditions the a posteriori probability computing multireceiver has been derived (Ref. 4 and 13). For each transmission the measuring equipment computes the multichannel gain and phase characteristic. These computations are announced to the detection structure of the multireceiver for use in detecting that particular transmission. In this respect the coherent termination may be viewed as an adaptive multireceiver, and the error rate derived for this system is valid for ideal measurement techniques. Adaptive as used here means a multireceiver which is capable of performing measurements on the transmission medium while simultaneously transmitting information and adjusting its modes of operation so as to optimize its performance with respect to some a priori chosen performance criterion. As prescribed by the likelihood function, and under the foregoing assumptions, the optimum coherent multireceiver passes the observed data from each channel output into its own receptor, as illustrated in Fig. 2a and 2b. For each channel output the appropriate phase measurement is inserted. Electronically speaking this may also be done in other ways. Following the phase insertion equipment is a filter matched to the signals stored at the transmitter. (The phantom lines in Fig. 2a and 2b denote both the other branches of the kth receiver.) The matched filter outputs are then weighted in accordance with the gain of the ith channel and sampled at the end of the signal interval. After phase shifting, filtering, weighting, and sampling, the outputs of all filters are summed and compared. The decision is then made in favor of that signaling waveform which gave rise to the largest summed multifilter output. This process is equivalent to the combining technique outlined by Brennan (Ref. 14).

The noncoherent multireceiver (see Fig. 2b) consists of filters matched to the signaling set $\{s_{ki}(t)\}$ $k = 1, 2, \cdots N$ and $i = 1, 2, \cdots M$. The output of these filters is followed by square-law envelope detectors which are sampled at the conclusion of the signaling interval, and the final decision is accomplished by comparing the sums of the samples from all receiver units. The larger sum determines the more likely transmitted signal. It can be shown (Ref. 4 and 6) that in the presence of independent Rayleigh fading in the multichannel, with fading sufficiently rapid that the amplitude of successive waveforms is approximately independent, this combining and decision method is the optimum a posteriori probability computer. Such is not the case if the multichannel contains fixed components. The difference is that the multireceiver nonlinear characteristic must change. Thus for



2. Diagram of the multireceivers

the Rician multichannel, the noncoherent multireceiver is actually suboptimum. For a discussion of the necessary nonlinear characteristic see Ref. 4.

D. Error Probability for the Coherent Multireceiver

For the coherent analysis we restrict the alphabet size to two signals,[6] i.e., $N = 2$. If we denote the observed data samples by $Q_k = (q_{k1}, q_{k2}, \cdots q_{kM})$, $k = 1, 2$, the multireceiver computes from these samples

$$\Delta Q = \sum_{i=1}^{M} a_i \,(q_{2i} - q_{1i}) \tag{6}$$

The resulting number ΔQ is compared at the termination of each baud with a threshold. If ΔQ exceeds this threshold, signal one is announced. On the other hand if ΔQ is below the threshold value, signal two is decided on.

Since the phase characteristic $\boldsymbol{\theta}$ is presumed to be known before reception of a baud we work with the probability density functions for the in-phase samples. This is given in Ref. 15, namely,

$$p_k(y) = \frac{1}{\sqrt{2\pi \sum_{i=1}^{M} a_i^2 \, E_i N_0 (1-\lambda)}} \times \exp\left[- \frac{\left[y - \sum_{i=1}^{M} (-1)^k \, a_i^2 \, E_i N_0 (1-\lambda) \right]^2}{\sum_{i=1}^{M} 2a_i^2 \, E_i N_0 (1-\lambda)} \right] \tag{7}$$

where a_i is the instantaneous voltage gain of the ith channel, M is the multichannel order, and λ is the normalized signal correlation coefficient. Using Eq. (7) it is possible to write the conditional error rate for the binary case as (see Ref. 13)

$$P_E(a_1, a_2, \cdots a_M) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\sum_{i=1}^{M} a_i^2 R_i (1-\lambda)}}^{\infty} \exp\left[- \frac{x^2}{2} \right] dx \tag{8}$$

where $R_i = \dfrac{E_i}{N_0}$. If we now let $X = \sum_{i=1}^{M} a_i^2$ and $E_i = E_j$ for all i and j, Eq. (8) becomes[7]

$$P_E(X) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{XR(1-\lambda)}}^{\infty} \exp\left[- \frac{x^2}{2} \right] dx \tag{9}$$

The probability density function associated with the gain of the ith propagation mode may be obtained from Eq. (5) by integrating over the phase variable θ_i. This yields

$$p(a_i) = \frac{a_i}{\sigma^2} \exp\left[- \frac{a_i^2 + \alpha_i^2}{2\sigma^2} \right] I_0 \left(\frac{\alpha_i a_i}{\sigma^2} \right); \; 0 \leq a_i \leq \infty$$

$$= 0 \text{ elsewhere} \tag{10}$$

where $I_0(x)$ is the modified Bessel function of zero order. In Appendix A it is shown that the density function $p(X)$ is given by

$$p(X) = \frac{1}{2\sigma^2} \left(\frac{X}{P} \right)^{\frac{M-1}{2}} \exp\left[- \frac{X + P}{2\sigma^2} \right] I_{M-1} \left(\frac{\sqrt{PX}}{\sigma^2} \right) \quad 0 \leq X \leq \infty$$

$$= 0 \text{ elsewhere} \tag{11}$$

where

$$P = \sum_{i=1}^{M} \alpha_i^2$$

and $I_{M-1}(x)$ is the modified Bessel function of order $M - 1$. The average error rate may be obtained by averaging Eq. (9) over all X, i.e.,

$$P_E(M) = \int_0^{\infty} p(X) \, P_E(X) \, dX \tag{12}$$

Substituting Eq. (8) and (11) into Eq. (12), letting $X = \sigma^2 t^2$, and rearranging, gives

$$P_E(M) = \int_0^{\infty} \frac{dt}{\sqrt{2\pi}} \int_{\sqrt{dt^2}}^{\infty} t \left[\frac{t}{a} \right]^{M-1} \exp\left[- \frac{t^2 + a^2 + x^2}{2} \right] \times I_{M-1}(at) \, dx \tag{13}$$

where $a = \sqrt{P}/\sigma$ and $d = \sigma^2 R(1\text{-}\lambda)$. Since the integrand is an absolutely integrable function, interchanging the order of integration in Eq. (13) is justified. Employing this procedure and adjusting the limits of integration yields

[6] Analysis for the N-ary alphabet may be performed; however, to make the analysis tractable, the correlation matrix for the signal set $\{s_{ki}(t)\}$ must be specified by Eq. (1). For orthogonal signals the problem is rather easy.

[7] At this point it appears that all M transmitter energy outputs for each signal must be assumed equal; otherwise the process of averaging over the a_i's appears to be formidable. See Section III, Conclusions.

$$P_E(M) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left[-\frac{x^2}{2}\right] dx \int_{\frac{x}{\sqrt{d}}}^\infty t \left[\frac{t}{a}\right]^{M-1} \times \exp\left[-\frac{t^2+a^2}{2}\right] I_{M-1}(at)\, dt \qquad (14)$$

Now the integration of Eq. (14) with respect to the variable t is the generalization of the Marcum Q-function (see Ref. 16 and 17) defined by

$$Q_M(\alpha,\beta) = Q(\alpha,\beta) + \exp\left[-\frac{\alpha^2+\beta^2}{2}\right] \sum_{k=1}^{M-1} \left(\frac{\beta}{\alpha}\right)^k I_k(\alpha\beta) \qquad (15)$$

where

$$Q(\alpha,\beta) = \int_\beta^\infty z \exp\left[-\frac{z^2+\alpha^2}{2}\right] I_0(\alpha z)\, dz \qquad (16)$$

Thus Eq. (14) may be written as

$$P_E(M) = \frac{1}{2}\left[1 - \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty Q_M\left(a, \frac{x}{\sqrt{d}}\right) \exp\left[-\frac{x^2}{2}\right] dx\right] \qquad (17)$$

We shall integrate Eq. (13) later; however, it is convenient to point out several special cases included in Eq. (17). First we assume $M = 1$ and Eq. (17) becomes[8]

$$P_E(1) = \frac{1}{2}\left[1 - \sqrt{\frac{2}{\pi}} \int_0^\infty Q\left(a, \frac{y}{\sqrt{d}}\right) \exp\left[-\frac{y^2}{2}\right] dy\right] \qquad (18)$$

This is the error rate for a coherent correlation receiver operating in the face of additive white Gaussian noise and Rician fading. If we assume further that the channel is completely random, i.e., $a = 0$, Eq. (18) reduces to a previous result (see Ref. 15):

$$P_E(1) = \frac{1}{2}\left[1 - \sqrt{\frac{d}{1+d}}\right] \qquad (19)$$

[8]This integral may be expressed in terms of the Q-function, i.e.,

$$P_E(1) = Q(a,b) - \frac{1}{2}\left[1 + \sqrt{\frac{d}{1+d}}\right] \exp\left[-\frac{\gamma^2(1+2d)}{2(1+d)}\right] \times I_0\left(\frac{\gamma^2}{2(1+d)}\right)$$

where

$$a^2 = \frac{\gamma^2[1 + 2d - 2\sqrt{d(1+d)}]}{2(1+d)}$$

$$b^2 = \frac{\gamma^2[1 + 2d + 2\sqrt{d(1+d)}]}{2(1+d)}$$

where $d = \sigma^2 R(1-\lambda)$. If we let $\lambda = 0$ in Eq. (18) we have a result derived by Montgomery (Ref. 18). On the other hand, if we let $\sigma = 0$, i.e., no fading, we get the familiar result (Ref. 9)

$$P_E(1) = \frac{1}{2}\left[1 - \operatorname{erf}\left(\sqrt{\frac{\alpha^2 R(1-\lambda)}{2}}\right)\right] = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{\alpha^2 R(1-\lambda)}{2}}}^\infty \exp[-z^2]\, dz \qquad (20)$$

Returning to the integration problem, we may rewrite Eq. (17) by using Eq. (15), as

$$P_E(M) = P_E(1) - \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{M-1} \left(\frac{\sqrt{c}}{a}\right)^k \exp\left[-\frac{a^2}{2}\right] \times \int_0^\infty y^k \exp\left[-y^2\left(\frac{c+1}{2}\right)\right] I_k(a\sqrt{c}\,y)\, dy \qquad (21)$$

where we let $c = 1/d$. Upon recognizing the fact that the first $M - 2$ term in the sum plus $P_E(1)$ is merely the error rate for $M - 1$ propagation modes $P_E(M-1)$, we may write Eq. (16) as

$$P_E(M) = P_E(M-1) - \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{c}}{a}\right)^{M-1} \exp\left[-\frac{a^2}{2}\right] \times \int_0^\infty y^{M-1} \exp\left[-y^2\left(\frac{c+1}{2}\right)\right] I_{M-1}(a\sqrt{c}\,y)\, dy \qquad (22)$$

In Appendix B it is shown that Eq. (22) reduces to

$$P_E(M) = P_E(M-1) - \frac{1}{2}\binom{2M-2}{M-1}\sqrt{\frac{d}{1+d}} \times \left[\frac{1}{4+4d}\right]^{M-1} \exp\left[-\frac{Ld}{1+d}\right] F\left(\frac{1}{2}, M, -\frac{L}{1+d}\right) \qquad (23)$$

where

$$L = \frac{P}{2\sigma^2} = \sum_{i=1}^M \gamma_i^2$$

$$d = \sigma^2 R(1-\lambda)$$

and $F(a, c, z)$ is the confluent hypergeometric function defined by

$$F(a,c,z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots \qquad (24)$$

In order to interpret Eq. (23) in terms of basic communication parameters it is convenient to define[9]

$$\beta = 2\sigma^2 R = \frac{2\sigma^2 E}{N_0}, \rho_i = \alpha_i^2 R$$

$$L\beta = \sum_{i=1}^{M} \rho_i, \mu = \frac{\beta(1-\lambda)}{2+\beta(1-\lambda)}, d = \frac{\beta}{2}(1-\lambda) \tag{25}$$

where β is physically related to the output signal-to-noise ratio produced by the random or scatter channel component and ρ_i is physically related to the output signal-to-noise ratio of the specular component existing in the ith channel. Thus in terms of these parameters Eq. (23) becomes

$$P_E(M) = P_E(M-1) - \frac{\sqrt{\mu}}{2}\binom{2M-2}{M-1}\left[\frac{1}{4+2\beta(1-\lambda)}\right]^{M-1} \times \exp[-L\mu]\, F\left(\frac{1}{2}, M, -\frac{2L}{2+\beta(1-\lambda)}\right) \tag{26}$$

for $M > 1$.

For the completely random multichannel, $\rho_i = 0$ for all $m = 1, 2, \cdots M$ and Eq. (26) reduces to a previously derived result (see Ref. 15):

$$P_E(M) = P_E(M-1) - \frac{1}{2}\sqrt{\mu}\binom{2M-2}{M-1} \times \left[\frac{1}{4+2\beta(1-\lambda)}\right]^{M-1} \tag{27}$$

for $M > 1$, while for $M = 1$, Eq. (19) is the solution.

Now Eq. (23) may be regarded as a difference equation for which the solution is easily shown to be

$$P_E(M) = P_E(1) - \frac{1}{2}\sqrt{\frac{d}{1+d}}\exp\left[-\frac{Ld}{1+d}\right] \times \sum_{m=2}^{M}\binom{2m-2}{m-1}\left(\frac{1}{4+4d}\right)^{M-1} F\left(\frac{1}{2}, m, -\frac{L}{1+d}\right) \tag{28}$$

where $P_E(1)$ is defined by Eq. (18). For the completely random multichannel we have

$$P_E(M) = P_E(1) - \frac{1}{2}\sqrt{\frac{d}{1+d}}\sum_{m=2}^{M}\binom{2m-2}{m-1}\left(\frac{1}{4+4d}\right)^{M-1} \tag{29}$$

where $P_E(1)$ is defined by Eq. (18). If the propagation modes are fixed and $a_i = \alpha_i$ for all $i = 1, 2, \cdots M$, Eq. (8) represents the multilink performance.

We now return to the problem of integrating Eq. (14) exactly. By utilizing a result obtained by Rice (Ref. 10), namely,

$$\int_0^x t\left[\frac{t}{a}\right]^{v-1}\exp\left[-\frac{t^2+a^2}{2}\right] I_{v-1}(at)dt = \exp\left[-\frac{x^2+a^2}{2}\right]\sum_{m=0}^{\infty}\left(\frac{x}{a}\right)^{v+m} I_{v+m}(ax) \tag{30}$$

it is possible to obtain an expression not involving integrals for the coherent multireceiver error rate. Substitution of Eq. (30) into Eq. (14) and performing the necessary integration leads to (see Appendix B)

$$\begin{aligned} P_E(M) &= \frac{1}{2}\sqrt{\frac{d}{1+d}}\exp\left[-\frac{Ld}{1+d}\right]\sum_{k=M}^{\infty}\binom{2k}{k} \times \left(\frac{1}{4+4d}\right)^k F\left(\frac{1}{2}, k+1, -\frac{L}{1+d}\right); d > 0 \\ &= \frac{1}{2}; d = 0 \end{aligned} \tag{31}$$

where d and L are defined by Eq. (25). As a special case, with $M = 1$, Eq. (31) becomes

$$P_E(1) = \frac{1}{2}\sqrt{\frac{d}{1+d}}\exp\left[-\frac{\gamma^2 d}{1+d}\right]\sum_{k=1}^{\infty}\binom{2k}{k} \times \left(\frac{1}{4+4d}\right)^k F\left(\frac{1}{2}, k+1, -\frac{\gamma^2}{1+d}\right) \tag{32}$$

for the single-path Rician fading channel. If $\gamma = 0$, we have

$$\begin{aligned} P_E(1) &= \frac{1}{2}\sqrt{\frac{d}{1+d}} - \sum_{k=1}^{\infty}\binom{2k}{k}\left(\frac{1}{4+4d}\right)^k; d > 0 \\ &= \frac{1}{2}; d = 0 \end{aligned} \tag{33}$$

Since

$$\binom{2k}{k} = 2^{2k}(-1)^k\binom{-\frac{1}{2}}{k}$$

[9]This definition of β is the same as that used by Turin (Ref. 4).

we may write

$$\sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{1}{4+4d}\right)^k = \sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\left[\frac{4}{4+4d}\right]^k$$
$$= \sqrt{\frac{d+1}{d}} \qquad (34)$$

Thus Eq. (33) becomes

$$P_E(1) = \frac{1}{2}\sqrt{\frac{d}{1+d}}\left[\sqrt{\frac{1+d}{d}} - 1\right]$$
$$= \frac{1}{2}\left[1 - \sqrt{\frac{d}{1+d}}\right] \qquad (35)$$

which agrees with Eq. (19), as it should.

E. Error Probability for the N-ary Noncoherent Multireceiver

We now derive the performance of the N-ary noncoherent multireceiver where the signal, chosen for transmission, is selected from a set of N equal-energy equiprobable orthogonal waveforms.

In this case we have available M channels and N signals and we assume, without loss of generality, that signal one was transmitted. Thus, at the output of the mth receiver unit, signal one has a normalized detected output x_m whose probability density function (pdf) may be shown to be[10]

$$p(x_m) = \frac{x_m}{1+\beta}\exp\left[-\frac{x_{1m}^2+\Delta_m}{2(1+\beta)}\right]$$
$$\times I_0\left(\sqrt{\frac{\Delta_m}{(1+\beta)^2}}\,x_{1m}\right);\ x_m > 0$$
$$= 0 \text{ elsewhere} \qquad (36)$$

where $\Delta_m = 2\rho_m = 2\alpha_m^2 R$ and $\beta = 2\sigma^2 R$. The other $N-1$ signal outputs from the multichannel have only noise samples y_{km} with pdf's given by

$$p(y_{km}) = y_{km}\exp\left[-\frac{y_{km}^2}{2}\right];\ y_{km} \geq 0$$
$$= 0 \text{ elsewhere} \qquad (37)$$

for $k = 2, 3, \cdots N$. The pdf's of $x_{1m}^2/2$ and $y_{km}^2/2$ are respectively

$$p_1(x_m) = \frac{1}{(1+\beta)}\exp\left[-\frac{x_m+\rho_m}{1+\beta}\right]$$
$$\times I_0\left(\sqrt{\frac{4\rho_m x_m}{(1+\beta)^2}}\right);\ x_m > 0$$
$$= 0 \text{ elsewhere}$$
$$p_k(y_m) = \exp\left[-y_m\right];\ y_m > 0$$
$$= 0 \text{ elsewhere} \qquad (38)$$

for $m = 1, 2, \cdots M$ and $k = 2, 3, \cdots N$. Moreover, the density function governing the summed signal statistics

$$X = \sum_{m=1}^{M} x_m$$

may be shown to be[10]

$$p_1(X) = \frac{1}{1+\beta}\left[\frac{X}{L\beta}\right]^{\frac{M-1}{2}}\exp\left[-\frac{X+L\beta}{1+\beta}\right]$$
$$\times I_{M-1}\left(\sqrt{\frac{L\beta X}{(1+\beta)^2}}\right);\ X > 0$$
$$= 0 \text{ elsewhere} \qquad (39)$$

where L and β are defined by Eq. (25) and $I_{M-1}(x)$ is the modified Bessel function of order $M-1$. The statistics for the noise variable

$$Y = \sum_{m=1}^{M} y_m$$

may be obtained from Eq. (39) by letting $L = \beta = 0$ and using the series expansion for $I_{M-1}(x)$. This turns out to be

$$p_k(Y) = \frac{Y^{M-1}}{\Gamma(M)}\exp\left[(-Y)\right] \qquad (40)$$

for $k = 2, 3, \cdots N$. Equations (39) and (40) provide the necessary information for computing the error date. To find the error probability we calculate the probability of correct detection, i.e., the probability that $X > Y$ for all $k = 2, 3, \cdots N$, and then average over all possible X. Hence, the probability of correct detection may be written as[11]

$$P_c(N,M) = \int_0^{\infty} p_1(X)\left[\int_0^{X} p_k(Y)dY\right]^{N-1} dX \qquad (41)$$

Substituting Eq. (40) into Eq. (41), and integrating, yields

$$P_c(N,M) = \int_0^{\infty} p_1(X)\left[1 - \sum_{m=0}^{M-1}\frac{X^m}{m!}\exp(-X)\right]^{N-1} dX \qquad (42)$$

[10] In *Probability Distribution for Practical Applications*, by W. C. Lindsey, Technical Report No. 32-511, Jet Propulsion Laboratory, Pasadena, Calif. (in preparation).

[11] In general the product notation should be used; in this case, however, $p_k(Y)$ is independent of k.

which becomes, upon using the binomial theorem,

$$P_c(N,M) = \sum_{n=0}^{N-1} (-1)^n \binom{N-1}{n} \int_0^\infty p_1(X) \left[\sum_{n=0}^{M-1} \frac{X^n}{n!} e^{-X}\right]^n dx \tag{43}$$

The multinominal theorem may be used to expand and integrate Eq. (43); however, the end result, for practical values of M and N, breaks up into a formidably large number of terms. A more tractable general procedure would be to evaluate the error probability directly from the probability distributions $p_1(X)$ and $p_k(Y)$ by means of numerical integration on a general-purpose computer. If we let $\rho_m = 0$ for all $m = 1, 2, \cdots M$, we have a result which agrees with Hahn (Ref. 19).

For large N it is appropriate to derive asymptotic formulas which are valid for $N \rightarrow \infty$. To do so, we approximate

$$G(X) = \left[1 - \sum_{m=0}^{M-1} \frac{X^m}{m!} e^{-X}\right]^{N-1} = \begin{cases} 0; X < X_0 \\ 1; X < X_0 \end{cases} \tag{44}$$

by the unit step occurring at $X = X_0$, where X_0 is defined by the requirement that for $X = X_0$ the exact function equals

$$G(X_0) = \left[1 - \sum_{m=0}^{M-1} \frac{X_0^m}{m!} e^{-X_0}\right]^{N-1} = \frac{1}{2} \tag{45}$$

Thus for large N we have to a good approximation

$$P_c(N,M) = \int_{X_0}^\infty p_1(X)\, dX \tag{46}$$

$$P_c(N,M) = Q_M\left[\sqrt{\frac{L\beta}{1+\beta}}, X_0\right] \tag{47}$$

where $Q_M(\alpha,\beta)$ is defined by Eq. (15). Interestingly enough, Eq. (47) says that for large code sizes, the probability of correct detection is the same as the probability of correctly detecting M signals by means of a threshold detection system. The threshold value is determined by Eq. (45).

If we return to Eq. (43), let $M = 1$ (single-channel reception), and note that $P_E(N) = 1 - P_c(N,1)$, we find that after integrating

$$P_E(N) = \sum_{n=1}^{N-1} \frac{(-1)^{n+1}}{n+1+n\beta} \binom{N-1}{n} \exp\left[-\frac{n\gamma^2\beta}{n+1+n\beta}\right] \tag{48}$$

where $\binom{N-1}{n}$ is the binomial coefficient and $\gamma^2\beta = \rho$.

Letting $\beta = 0$ in Eq. (48) we obtain[12]

$$P_E(N,1) = \sum_{n=1}^{N-1} \frac{(-1)^{n+1}}{n+1} \binom{N-1}{n} \exp\left[-\frac{n\rho}{n+1}\right] \tag{49}$$

which agrees with Reiger's result (Ref. 20). For the completely random channel we let $\beta = 0$ in Eq. (47) and find that[13]

$$P_E(N,1) = \sum_{n=1}^{N-1} \binom{N-1}{n} \frac{(-1)^{n+1}}{n+1+n\beta} \tag{50}$$

Upon letting $N = 2$ in Eq. (49) and (50) we have results that agree with those derived by Reiger (Ref. 21) and Turin (Ref. 4 and 9).

F. *Error Rate for the Noncoherent Binary Case (N = 2)*

In the binary case we write the error rate as

$$P_E(M) = \int_0^\infty p_1(X)\, dX \int_X^\infty p_2(Y)\, dY \tag{51}$$

Substituting Eq. (40) into Eq. (51) and integrating yields

$$P_E(M) = \sum_{m=0}^{M-1} \int_0^\infty p_1(X) \frac{X^m}{m!} e^{-X} dX \tag{52}$$

whereupon using Eq. (39) in Eq. (52) gives, after rearranging,

$$P_E(M) = \sum_{m=0}^{M-1} \frac{1}{m!(1+\beta)} \left(\frac{1}{L\beta}\right)^{\frac{M-1}{2}} \exp\left[-\frac{L\beta}{1+\beta}\right] \times \int_0^\infty X^{\frac{M-1}{2}+m} \exp\left[-\left(\frac{2+\beta}{1+\beta}\right)X\right] \times I_{M-1}\left(\sqrt{\frac{4L\beta X}{(1+\beta)^2}}\right) dX \tag{53}$$

If we now let $Y^2 = X$ it may be shown (see Appendix C) that Eq. (53) integrates to

$$P_E(M) = \left[\frac{1}{2+\beta}\right]^M \exp\left[-\frac{L\beta}{2+\beta}\right] \sum_{m=0}^{M-1} \binom{M+m-1}{m} \times \left(\frac{1+\beta}{2+\beta}\right)^m F\left(-m, M, \frac{-L\beta}{(1+\beta)(2+\beta)}\right) \tag{54}$$

[12] Turin (Ref. 22) has also arrived at this result and has evaluated the error probability for several values of N and ρ.

[13] Numerical results for Eq. (48) and (50) are presently being computed.

where $F(a,c,z)$ is the confluent hypergeometric function defined by Eq. (24) and

$$L = \sum_{i=1}^{M} \gamma_i^2$$

If we let $\gamma_i^2 = 0$ for all $i = 1, 2, \cdots M$, Eq. (54) reduces to results derived by Turin (Ref. 5) and Pierce (Ref. 6) for the completely random multichannel, i.e.,

$$P_E(M) = \left[\frac{1}{2+\beta}\right]^M \sum_{m=0}^{M-1} \binom{M+m-1}{m} \left(\frac{1+\beta}{2+\beta}\right)^m \quad (55)$$

On the other hand, for the fixed-mode multichannel we let $\beta = 0$ and find that

$$P_E(M) = \left[\frac{1}{2}\right]^M \exp\left[-\sum_{i=1}^{M} \rho_i\right] \sum_{m=0}^{M-1} \binom{M+m-1}{m} \left(\frac{1}{2}\right)^m \times F\left(-m, M, \sum_{i=1}^{M} \frac{-\rho_i}{2}\right) \quad (56)$$

It is convenient to write Eq. (54) in terms of the generalized Laguerre polynomials $L_n^\alpha(x)$ (Ref. 23) defined by

$$L_n^\alpha(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!} \quad (57)$$

where

$$\binom{a}{b} = \frac{a!}{b!\,(a-b)!}$$

is the binomial coefficient. Conveniently enough, the generalized Laguerre polynomials are related to the confluent hypergeometric function (Ref. 23) by the relationship

$$F(-n, \alpha+1, z) = \frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} L_n^\alpha(z) \quad (58)$$

Upon substituting Eq. (58) into Eq. (54) and simplifying, we obtain

$$P_E(M) = \left[\frac{1}{2+\beta}\right]^M \exp\left[-\frac{L\beta}{2+\beta}\right] \sum_{m=0}^{M-1} \left(\frac{1+\beta}{2+\beta}\right)^m \times L_m^{M-1}\left[\frac{-L\beta}{(1+\beta)(2+\beta)}\right] \quad (59)$$

If we now substitute Eq. (57) into Eq. (59) we have the final result, namely,

$$P_E(M) = \left[\frac{1}{2+\beta}\right]^M \exp\left[\frac{-L\beta}{2+\beta}\right] \sum_{m=0}^{M-1} \cdot \sum_{n=0}^{m} \binom{m+M-1}{m-n} \times \left(\frac{1+\beta}{2+\beta}\right)^m \frac{x^n}{n!} \quad (60)$$

where

$$x = \frac{L\beta}{(1+\beta)(2+\beta)}$$

G. *Asymptotic Characteristics of the Coherent Multireceiver for Various Multichannel Conditions*[14]

The asymptotic performance characteristics presented in this Section describe system performance at low probabilities of committing an error. These results are derived in Appendix B.

The approach is to present the result for the completely random multichannel and then proceed through the other results, presuming that the multichannel characteristics change. This displays the rapidity with which the performance of the coherent multireceiver changes as the propagation characteristics change.

For low error probabilities, i.e., $\beta >> 1$, we have for the completely random multichannel supporting equally reliable propagation modes[15]

$$P_{E_1}(M; \boldsymbol{\theta}, \mathbf{a}, \boldsymbol{\tau}) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{2\beta(1-\lambda)}\right]^M \quad (61)$$

The quantity

$$\binom{2M}{M}$$

is the binomial coefficient

$$\frac{(2M)!}{(M!)^2}$$

From this expression it is obvious that phase-reversal binary signaling techniques yield a 3-db improvement in signal-to-noise ratio over ordinary orthogonal frequency-shift-keyed signaling techniques. This is not too surprising since we have assumed that coherent reception is possible. If we let $\lambda = 0$, the result agrees with that obtained by Pierce (Ref. 6).

If we now consider the Rician fading multichannel with small specular components, i.e., $\beta > \rho$, it is possible to show that (see Appendix B)

$$P_{E_2}(M; \boldsymbol{\theta}, \mathbf{a}, \boldsymbol{\tau}) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{2\beta(1-\lambda)}\right]^M \prod_{i=1}^{M} \exp\left[-\gamma_i^2\right] \quad (62)$$

[14]For a more detailed discussion of the results given in Sections II-G, II-H, and II-I, see Ref. 24.

[15]The subscript n signifies the type of multichannel under consideration. If $n = 1$, we mean the completely random multichannel, $n = 2$ signifies the Rician fading multichannel, and $n = 3$ signifies the fixed-mode multichannel. This notation is employed in Sections II-G, II-H, and II-I.

which agrees with Eq. (61) if we let $\gamma_i^2 = 0$ for all $i = 1, 2, \cdots M$. (Note here that if N of the propagation modes are Rayleigh-distributed then $M-N$ are Rician-distributed. This is what we define as the mixed-mode multichannel.) The effect of multichannel specular components is to insert exponential factors in the system performance expression, whereas the random nature of the multichannel introduces inverse factors, i.e., factors which decrease with increasing signal-to-noise ratio of the random component. These exponential factors may remarkably increase system performance. In particular, the ratio of Eq. (62) to Eq. (61) is

$$\frac{P_{E_2}(M;\boldsymbol{\theta},\mathbf{a},\boldsymbol{\tau})}{P_{E_1}(M;\boldsymbol{\theta},\mathbf{a},\boldsymbol{\tau})} = \prod_{i=1}^{M} \exp\left[-\gamma_i^2\right] \le 1 \qquad (63)$$

If the fixed-channel components are equally reliable we have

$$P_{E_2}(M;\boldsymbol{\theta},\mathbf{a},\boldsymbol{\tau}) = P_{E_1}(M;\boldsymbol{\theta},\mathbf{a},\boldsymbol{\tau}) \exp\left[-M\gamma^2\right] \qquad (64)$$

for $\beta > 1$.

We now assume that the multichannel state changes to the conditions where the fixed components are larger than the random components, i.e., $\rho > \beta$. In this case we have asymptotically (see Appendix B)

$$P_{E_2}(M;\boldsymbol{\theta},\mathbf{a},\boldsymbol{\tau}) \sim \frac{\prod_{i=1}^{M} \exp - \left[\gamma_i^2 \left(\frac{\beta(1-\lambda)}{2+\beta(1-\lambda)}\right)\right]}{\sqrt{2\pi \sum_{i=1}^{M} \rho_i (1-\lambda)}} \times \left[\frac{2}{2+\beta(1-\lambda)}\right]^{M-1} \qquad (65)$$

where $\gamma_i^2 \beta = \rho_i$. From Eq. (65) we note that the exponential factor is now the dominant feature in the asymptotic charactristic. The inverse factor due to the random channel component is still present, however.

As a final condition we assume that the multichannel modes become fixed. In this case $\beta = 0$ and we have for $\rho > 0$

$$P_{E_3}(M;\boldsymbol{\theta},\mathbf{a},\boldsymbol{\tau}) \sim \frac{\prod_{i=1}^{M} \exp\left[-\frac{\rho_i(1-\lambda)}{2}\right]}{\sqrt{2\pi \sum_{i=1}^{M} \rho_i (1-\lambda)}} \qquad (66)$$

The salient feature of this result is that system performance increases approximately exponentially with increasing signal-to-noise ratio. Recall from Eq. (61) that system performance for the completely random multichannel increased inversely with the Mth power of β whereas we observe the exponential behavior for the fixed-mode multichannel. If we let $M=1$ in Eq. (66) we have a result which agrees with that of Turin (Ref. 9). It is interesting to note that Eq. (66) is recognizable as the asymptotic expansion for the error function evaluated at

$$\sqrt{\sum_{i=1}^{M} \rho_i (1-\lambda)}$$

This provides a convenient check with the same result derived by another method in Appendix B.

H. *Asymptotic Characteristics of the Noncoherent Multireceiver for Various Multichannel Conditions*

We begin the discussion by assuming that the multichannel is completely random with the additional assumption that transmitted signals are orthogonal. System performance utilizing correlated signals has been derived; however, the asymptotic results have not been computed.

For the completely random multichannel we have, asymptotically,

$$P_{E_1}(M;\boldsymbol{\tau}) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{\beta}\right]^{M} \qquad (67)$$

for $\beta > 1$. The quantity

$$\binom{2M}{M}$$

is the binomial coefficient

$$\frac{(2M!)}{(M!)^2}$$

Pierce (Ref. 6) was the first to derive this result. System performance increases for a fixed M inversely with the signal-to-noise ratio of the random component. Note in Eq. (67) the absence of the exponential factors.

Suppose now, that due to changing propagation characteristics, the multichannel possesses small specular components of distinct strength α_i. For this situation we find for $\beta > \rho$ (see Appendix C)

$$P_{E_2}(M;\boldsymbol{\tau}) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{\beta}\right]^{M} \prod_{i=1}^{M} \exp\left[-\gamma_i^2\right] \qquad (68)$$

as the asymptotic result for the Rician multichannel. If we let $\gamma_i^2 = 0$ for all $i = 1, 2, \cdots M$, we have Eq. (11), which agrees with Pierce's result. From Eq. (68) we may conclude that the exponential factors are existent in the system error rate as a result of the specular channel components and the inverse factor is due to the random channel component. These exponential factors could remarkably increase system performance. Taking the ratio of Eq. (68) to Eq. (67) yields

$$\frac{P_{E_2}(M;\boldsymbol{\tau})}{P_{E_1}(M;\boldsymbol{\tau})} = \prod_{i=1}^{M} \exp\left[-\gamma_i^2\right] \leq 1 \qquad (69)$$

which is the same ratio obtained under similar conditions for the coherent case.

Assuming that further changes in the propagation characteristics introduce large specular components, we have established (see Appendix C) the following bound for $\beta < \rho$:

$$P_{E_1}(M;\boldsymbol{\tau}) < \left[\frac{1}{2+\beta}\right]^M \prod_{i=1}^{M} \exp\left[-\frac{\gamma_i^2\beta}{2+\beta}\left(1-\frac{1}{2+\beta}\right)\right] \qquad (70)$$

where $\gamma_i^2\,\beta = \rho_i$. For this multichannel condition the exponential factors dominate system performance, whereas the inverse factor appears to be the least significant.

As a final multichannel state we assume that the random components are zero; i.e., $\beta = 0$, or the propagation modes are fixed. The asymptotic result for this situation is

$$P_{E_3}(M;\boldsymbol{\tau}) \sim \frac{1}{2^M} \prod_{i=1}^{M} \exp\left[-\frac{\rho_i}{4}\right] \qquad (71)$$

Again the performance characteristic possesses only exponential factors as we would expect to find. For $M = 1$ we see that Eq. (71) is 3 db away from the exact curve.

I. Performance Comparison for the Two Multireceiver Terminations

Having described, for various forms of multichannel conditions, the asymptotic performance characteristics of the individual multireceivers it remains to consider the problem of system comparison.

We begin the comparison by considering the completely random multichannel. For convenience we rewrite Eq. (61) and (67), i.e., for $\beta > 1$

$$P_{E_1}(M;\boldsymbol{\theta},\mathbf{a},\boldsymbol{\tau}) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{2\beta(1-\lambda)}\right]^M \qquad (72)$$

$$P_{E_1}(M;\boldsymbol{\tau}) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{\beta}\right]^M \qquad (73)$$

Comparing these two expressions we see that the coherent multireceiver which performs measurements on the channel is capable of yielding, for any M, a 6-db improvement in signal-to-noise ratio over the noncoherent multireceiver. Moreover, it has been shown (Ref. 15) that for low signal-to-noise ratios the coherent multireceiver outperforms the noncoherent multireceiver by an 8-db factor (see also Fig. 3, 4, and 5). This corresponds to a considerable saving in transmitted power.

Assuming that the propagation medium introduces small specular components we write from Eq. (62) and (68)

$$P_{E_2}(M;\boldsymbol{\theta},\mathbf{a},\boldsymbol{\tau}) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{2+\beta(1-\lambda)}\right]^M \prod_{i=1}^{M} \exp\left[-\gamma_i^2\right] \qquad (74)$$

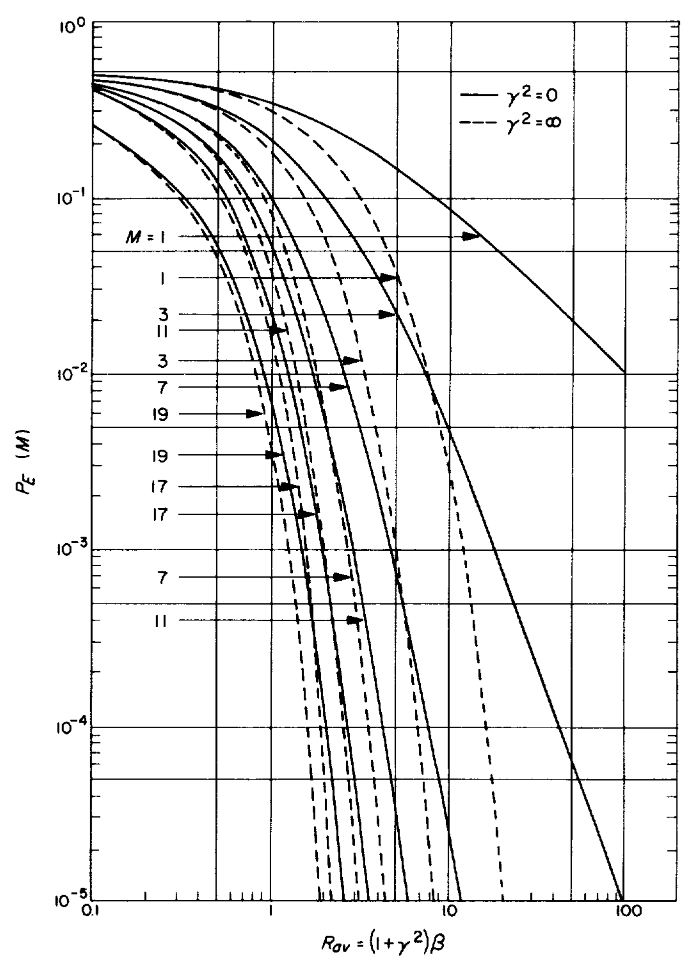


Fig. 3. Error probability for noncoherent multireceiver

$$P_{E_2}(M;\boldsymbol{\tau}) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{\beta}\right]^{M}\prod_{i=1}^{M}\exp\left[-\gamma_i^2\right] \tag{75}$$

The same conclusions may be reached for the mixed-mode multichannel as were pointed out for the completely random multichannel. However, for equivalent transmitter powers, system performance would be superior as a result of the multichannel specular components.

For the multichannel with large specular components we have from Eq. (65) and (70)

$$P_{E_2}(M;\boldsymbol{\theta},\mathbf{a},\boldsymbol{\tau}) \sim \frac{\prod_{i=1}^{M}\exp\left[-\gamma_i^2\left(\frac{\beta(1-\lambda)}{2+\beta(1-\lambda)}\right)\right]}{\sqrt{2\pi\sum_{i=1}^{M}\rho_i(1-\lambda)}} \times\left[\frac{2}{2+\beta(1-\lambda)}\right]^{M-1} \tag{76}$$

$$P_{E_2}(M;\boldsymbol{\tau}) < \left[\frac{1}{2+\beta}\right]^{M}\prod_{i=1}^{M}\exp\left[-\frac{\gamma_i^2\,\beta}{2+\beta}\left(1-\frac{1}{2+\beta}\right)\right] \tag{77}$$

where $\rho_i = \gamma_i^2\,\beta$. By comparing these two equations it is difficult to state the exact amount by which the coherent multireceiver outperforms the noncoherent multireceiver. The results, however, indicate important system performance trends and it is an intuitive conclusion that the coherent multireceiver would outperform the noncoherent multireceiver. The precise amount of improvement is the subject of the next Section.

J. Numerical and Graphical Results for Both the Coherent and Noncoherent Multireceivers

Plotted in Fig. 3, 4, and 6 is the noncoherent system error rate as a function of [16]

[16]Again we have adopted Turin's notation (Ref. 4). Our γ^2, however, is one-half the γ^2 defined in Ref. 4.

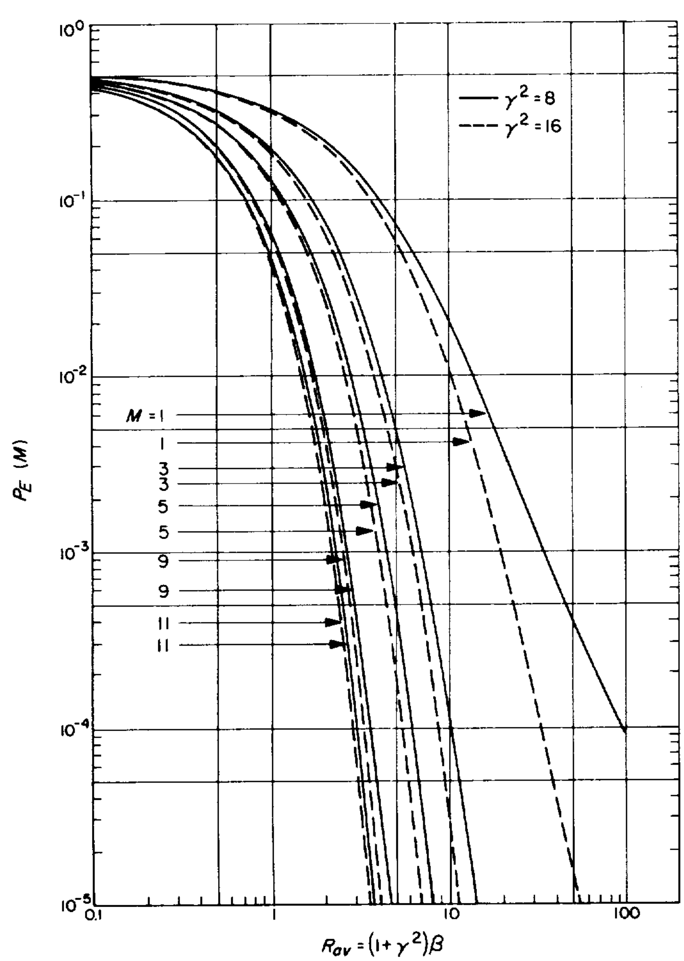


Fig. 4. Error probability for noncoherent multireceiver

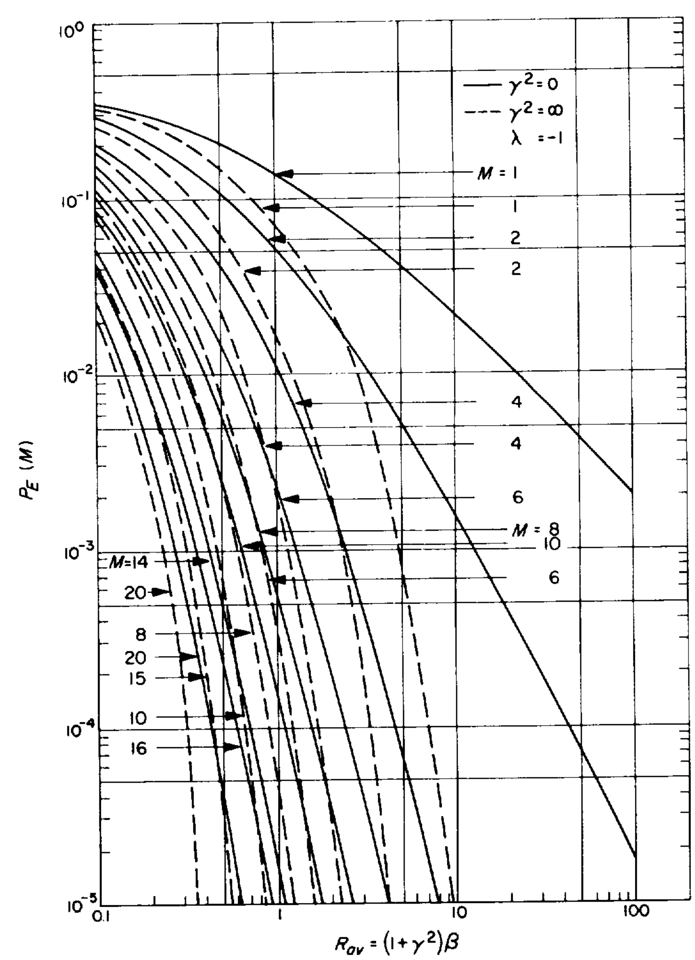


Fig. 5. Error probability for coherent multireceiver

$$R_{av} = \frac{E_{av}}{N_0} = \left(\frac{\alpha^2 + 2\sigma^2}{N_0}\right) E = (1 + \gamma^2)\beta \qquad (78)$$

which, as we have seen, is the ratio of the average received energy to the noise power density. For convenience of computation we have assumed that all propagation modes are equally reliable; i.e., the specular channel components are of equivalent strength. We have already assumed that the random components possess the same mean square value. Figures 3, 4, and 6 show the dependence of the probability of faulty reception on R_{av} for four values of the parameter $\gamma^2 = 0, 1, 8, 16$ and ∞, and for several multichannel orders. As seen from Fig. 3, 4, and 6, for any value of γ^2, multichannel reception increases the communication reliability as compared to single-channel reception, whereas the effectiveness of the multichannel order is greater as γ^2 becomes smaller. If we compare trichannel reception with single-channel reception for $\gamma^2 = 8, 16, \infty$ and for a probability of erroneous reception equal to 10^{-3}, we find a power gain approximately equal to 6.68, 5.0, and 3.67 db, respectively. For an error probability of 10^{-2} and $\gamma^2 = 0, 8, 16, \infty$, we realize a power gain of approximately 11.0, 5.0, 4.15, and 3.66 db. Furthermore, these curves allow us to estimate the extent to which the transmitted signal energy has to be increased to detect the signal with a certain reliability for the same multichannel order. We see that for large γ^2 and the same error probability, the required increase in transmitter energy is small and becomes smaller as M is increased. As a matter of fact, the required increase in transmitter energy is smaller for low signal-to-noise ratios than it is for large signal-to-noise ratios. This result is easily understandable from a physical point of view. When the signal-to-noise ratio is small the amplitude of the signal will sometimes be large, as a result of the fading, hence leading to the same signal-to-noise ratio that would occur if the signal amplitude were fixed. On the other hand, for large signal-to-noise ratios, the signal amplitude will sometimes be small due to its fluctuations, and consequently requires a large transmitter energy to make certain that on the average the signal will not fade below the required detection reliability.

In the coherent case, Eq. (8) and (27) have been evaluated on a digital computer, and the results for selected values of M and $\gamma^2 = 0, \infty$ are presented in Fig. 5. Again we see that the effect of multichannel reception is dependent on the value of γ^2. In particular, multichannel reception is more effective when applied to the completely random multichannel, i.e., $\gamma^2 = 0$, than when applied to the fixed-mode multichannel. As a function of the multichannel order M we see that large gains in signal power are realized for small values of M, whereas for $M > 6$ the gain in signal power is not as significant. Note that for large values of $M > 10$ the curves lie relatively close together. This tends to suggest that in the coherent case, multichannel reception becomes less effective for larger values of M regardless of the fading properties of the transmission medium. It appears that for large M, system performance is limited by the additive noise. Comparing dual-channel reception with single-channel reception for an error probability of 10^{-2} we realize, for $\gamma^2 = 0, \infty$, an approximate power gain of 8.35 and 3.67 db respectively. For an error rate of 10^{-3} we obtain a gain of approximately 11.7 and 3 db respectively.

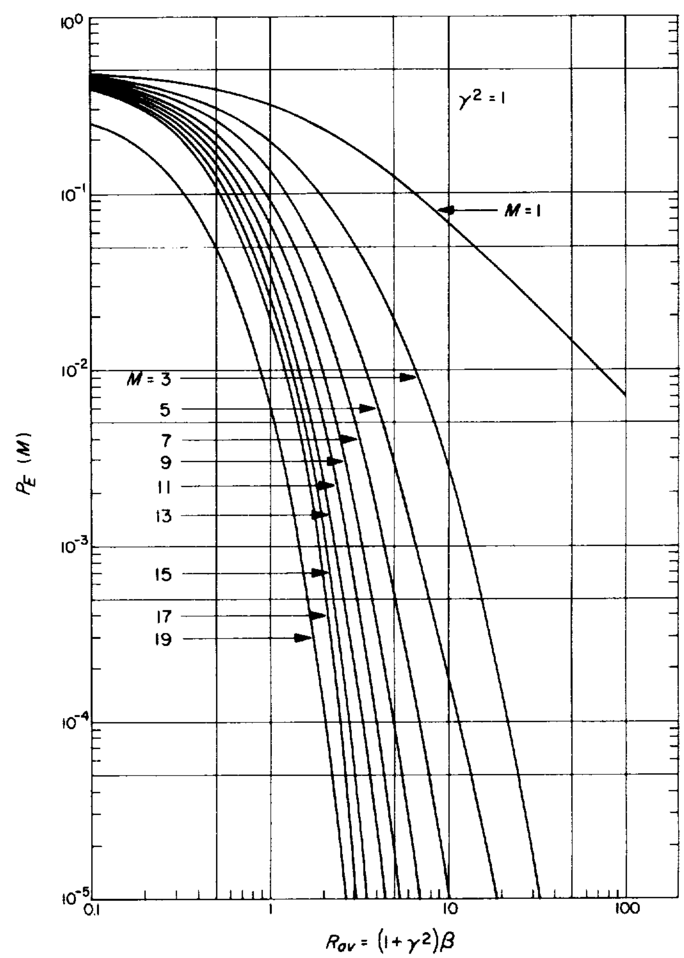


Fig. 6. Error probability for noncoherent multireceiver

III. CONCLUSIONS

The error probabilities have been derived and graphically illustrated for the coherent and noncoherent multilink communication systems which transmit information through the Rician fading multichannel. The effectiveness of multichannel reception is highly dependent on the basic multichannel parameter γ^2. In particular, multichannel reception is more effective when applied to multichannels supporting small γ^2's. The asymptotic error expressions indicate the relative gain in signal-to-noise ratio obtainable using the coherent rather than the noncoherent multireceiver termination. For large signal-to-noise ratios, 6 db is an upper bound on improvement, while for small signal-to-noise ratios approximately 8 db may be realized.

The results for the coherent case provide ideal performance characteristics for the newer types of data-transmission systems, i.e., systems which are designed to perform multichannel measurements. For the N-ary noncoherent multireceiver, it does not appear to be too difficult to use numerical integration techniques to obtain numerical results for the error rate.

In the foregoing analysis, it has been assumed that the energies associated with the signals stored at all M transmitters are equal. If, on the other hand, we had attempted to derive the asymptotic formulas under the assumption of distinct signal energies we would have encountered difficulty when attempting to average over the multichannel gain. This tends to suggest that there could exist an a priori signal-energy distribution which will minimize the error probability. One would expect to find that most of the energy should be transmitted into those channels which are not encountering severe or deep fades. This is in contrast to the fixed multichannel case where the asymptotic error probability depends only on the total transmitted energy. Price has noted this result (Ref. 1). The way in which the energies should be distributed among the various channels is yet to be determined. However, if the transmitter is unaware of the multichannel gain, intuitively the best procedure is to distribute the total energy uniformly.

APPENDIX A

By means of characteristic functions we wish to develop Eq. (11). We begin by writing the characteristic function of the variable a_i^2 from Foster and Campbell (Ref. 25), i.e.,

$$C_{a_i^2}(S) = \frac{1}{2\sigma^2} \exp\left[-\gamma_i^2\right]\left[\frac{1}{S + \frac{1}{2\sigma^2}}\right] \times \exp\left[\frac{\gamma_i^2}{2\sigma^2\left(S + \frac{1}{2\sigma^2}\right)}\right] \qquad \text{(A-1)}$$

The characteristic function for the variable X is the product of M expressions like Eq. (A-1). Hence

$$C_X(S) = \left(\frac{1}{2\sigma^2}\right)^M \cdot \left[\frac{1}{S + \frac{1}{2\sigma^2}}\right]^M \exp\left[-\frac{P}{2\sigma^2}\right] \times \exp\left[\frac{P}{4\sigma^4\left(S + \frac{1}{2\sigma^2}\right)}\right] \qquad \text{(A-2)}$$

where

$$P = \sum_{i=1}^{M} \alpha_i^2$$

The inverse transform of Eq. (A-2) is given by the Foster and Campbell Fourier transform pair 650.0 (Ref. 25). The result is Eq. (11), namely,

$$p(X) = \frac{1}{2\sigma^2}\left(\frac{X}{P}\right)^{\frac{M-1}{2}} \exp\left[-\frac{X+P}{2\sigma^2}\right] I_{M-1} \times \left(\frac{\sqrt{PX}}{\sigma^2}\right); 0 \leq X < \infty$$

$$= 0 \text{ elsewhere} \qquad \text{(A-3)}$$

APPENDIX B

1. *Derivation of Eq. (23)*

In order to derive Eq. (23) from Eq. (22) we need a result given by Watson (see Ref. 26, pp. 393–394), namely,

$$\int_0^\infty \exp\left[-a^2x^2\right] x^{\mu-1} I_v(bx)\, dx = \frac{b^v \Gamma\left(\frac{\mu+v}{2}\right)}{2^{v+1} a^{\mu+v} \Gamma(v+1)} \times \exp\left[\frac{b^2}{4a^2}\right] F\left(\frac{v-\mu}{2}+1, v+1, \frac{-b^2}{4a^2}\right) \qquad \text{(B-1)}$$

where $F(a,c,z)$ is the confluent hypergeometric function defined in the text by Eq. (24). Therefore

$$I(M) = \int_0^\infty y^{M-1} \exp\left[-y^2\left(\frac{1+c}{2}\right)\right] I_{M-1}(a\sqrt{c}\, y)\, dy$$

$$= \frac{(a\sqrt{c})^{M-1}\, \Gamma\left(M-\frac{1}{2}\right)}{2^M \left(\frac{1+c}{2}\right)^{(M-\frac{1}{2})} \Gamma(M)} \exp\left[\frac{a^2c}{2(1+c)}\right] \times F\left[\frac{1}{2}, M, \frac{-a^2c}{2(1+c)}\right] \qquad \text{(B-2)}$$

and from Eq. (22) we write

$$P_E(M) = P_E(M-1) - G(L,d,M) \qquad \text{(B-3)}$$

where

$$G(L,d,M) = \frac{1}{\sqrt{2\pi}}\left[\frac{\sqrt{c}}{a}\right]^{M-1} \exp\left[-\frac{a^2}{2}\right] I(M) \qquad \text{(B-4)}$$

Substituting Eq. (B-2) into Eq. (B-4) gives

$$G(L,d,M) = \frac{1}{2\sqrt{\pi}}\left(\frac{c}{1+c}\right)^{M-1}\sqrt{\frac{1}{1+c}}\,\frac{\Gamma(M-\frac{1}{2})}{\Gamma(M)} \times \exp\left[-\frac{a^2}{2}\left(\frac{1}{1+c}\right)\right]F\left[\frac{1}{2},M,-\frac{a^2c}{2(1+c)}\right] \quad \text{(B-5)}$$

Using the well-known result that

$$\Gamma\left(M-\frac{1}{2}\right) = \frac{(2M-2)!}{(M-1)!\,2^{2(M-1)}}\sqrt{\pi} \quad \text{(B-6)}$$

Eq. (B-5) becomes

$$G(L,d,M) = \frac{1}{2}\sqrt{\frac{1}{1+c}}\left[\frac{c}{4+4c}\right]^{M-1}\binom{2M-2}{M-1} \times \exp\left[-\frac{a^2}{2}\left(\frac{1}{1+c}\right)\right]F\left[\frac{1}{2},M-\frac{a^2c}{2(1+c)}\right] \quad \text{(B-7)}$$

where

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}$$

is the binomial coefficient. Noting that

$$c = \frac{1}{d} \text{ and } \frac{a^2}{2} = \frac{P}{2\sigma^2} = L = \sum_{i=1}^{M}\gamma_i^2$$

we may write Eq. (B-7) as

$$G(L,d,M) = \frac{1}{2}\sqrt{\frac{d}{1+d}}\binom{2M-2}{M-1}\left(\frac{1}{4+4d}\right)^{M-1} \times \exp\left[-\frac{Ld}{1+d}\right]F\left[\frac{1}{2},M\,\frac{-L}{1+d}\right] \quad \text{(B-8)}$$

This is the required result.

2. Derivation of Eq. (31)

Changing the order of integration in Eq. (13) we have

$$P_E(M) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left[\frac{-x^2}{2}\right]dx\int_0^{\frac{x}{\sqrt{d}}} t\left[\frac{t}{a}\right]^{M-1} \times \exp\left[-\frac{t^2+a^2}{2}\right]I_{M-1}(at)\,dt \quad \text{(B-9)}$$

which is equivalent to Eq. (14). Substituting Rice's result, namely Eq. (30), into Eq. (B-9) and rearranging the terms yields

$$P_E(M) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{a^2}{2}\right]\sum_{m=0}^{\infty}\int_0^\infty\left[\frac{-\sqrt{c}}{a}x\right]^{M+m} \times \exp\left[-x^2\left(\frac{1+c}{2}\right)\right]I_{M+m}(a\sqrt{c}\,x)\,dx \quad \text{(B-10)}$$

Performing the integration of Eq. (B-10) by use of Eq. (B-1) we find

$$P_E(M) = \frac{1}{2}\exp\left[-\frac{Ld}{1+d}\right]\sqrt{\frac{d}{1+d}}\sum_{k=M}^{\infty}\binom{2k}{k}\left(\frac{1}{4+4d}\right)^k \times F\left(\frac{1}{2},k+1,-\frac{L}{1+d}\right);\ d>0$$

$$= \frac{1}{2};\ d=0 \quad \text{(B-11)}$$

which is Eq. (31).

3. *Asymptotic Characteristics for the Coherent Case*

If we assume that the multichannel supports small specular components, i.e., $P<1$, then Eq. (A-3) reduces to

$$p(X) \sim \frac{X^{M-1}}{(2\sigma^2)^M\Gamma(M)}\exp\left[-L\right] \quad \text{(B-12)}$$

since for small values of X we have the approximation

$$I_{M-1}(X) \sim \frac{1}{\Gamma(M)}\left[-\frac{X}{2}\right]^{M-1} \quad \text{(B-13)}$$

The average error rate may be obtained from Eq. (9) by averaging over the variable X, i.e.,

$$P_E(M) = \int_0^\infty p(X)\,P_E(X)\,dX \quad \text{(B-14)}$$

where we have omitted the superscript notation used in the main text. Using Eq. (9) and (B-12) in (B-14) we may write

$$P_E(M) \sim \frac{1}{\sqrt{\pi}}\int_0^\infty p(X)\,dX\int_A^\infty \exp\left[-y^2\right]dy \quad \text{(B-15)}$$

where

$$A = \sqrt{\frac{XR(1-\lambda)}{2}}$$

$$R = \frac{E}{N_0} = \frac{\text{signal energy}}{\text{double-sided noise power density}}$$

λ = signal cross-correlation coefficient

If we now change orders of integration (which is justifiable), Eq. (B-15) becomes

$$P_E(M) \sim \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left[-y^2\right] dy \int_0^{\frac{y^2}{W}} p(X)dX \qquad \text{(B-16)}$$

where

$$W = \sqrt{R\left(\frac{1-\lambda}{2}\right)}$$

Substituting Eq. (B-12) into Eq. (B-16) yields

$$P_E(M) \sim \frac{\exp(-L)}{M!\,(2\sigma^2)^M \sqrt{\pi}} \int_0^\infty \frac{y^{2M}}{W^M} \exp\left[-y^2\right] dy \qquad \text{(B-17)}$$

This easily integrates with respect to y, giving

$$P_E(M) \sim \frac{\exp\left[-L\right]}{2(M!)} \frac{\left[1\cdot3\cdot5\cdots(2M-1)\right]}{\left[2\sigma^2 R\,(1-\lambda)\right]^M} \qquad \text{(B-18)}$$

which may be rewritten as

$$P_E(M) \sim \frac{\exp\left[-L\right](2M-1)!}{M!\,(M-1)!}\left[\frac{1}{2\beta(1-\lambda)}\right]^M \qquad \text{(B-19)}$$

or, more conveniently,

$$P_E(M) \sim \frac{1}{2}\binom{2M}{M} \exp\left[-L\right]\left[\frac{1}{2\beta\,(1-\lambda)}\right]^M \qquad \text{(B-20)}$$

which is Eq. (62) when we substitute for L. Upon letting $L = 0$ we have Eq. (61).

Consider now the multichannel which possesses small random components and large specular components. First we rewrite $p(X)$ in terms of the new variable $X = \sigma^2 t^2$. Thus $p(X)$ becomes

$$p(t) = t\left[\frac{t}{a}\right]^{M-1} \exp\left[-\frac{t^2+a^2}{2}\right] I_{M-1}(at)\,dt \qquad \text{(B-21)}$$

where

$$a = \frac{\sqrt{P}}{\sigma}$$

Hence Eq. (9) becomes

$$P_E(t) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{dt^2}}^\infty \exp\left[-\frac{x^2}{2}\right] dx \qquad \text{(B-22)}$$

where $d = \sigma^2 R(1-\lambda)$. Using the asymptotic expansion for the error function we have

$$P_E(t) \sim \frac{1}{\sqrt{2\pi dt^2}} \exp\left[-\frac{dt^2}{2}\right] \qquad \text{(B-23)}$$

for $dt^2 > 0$. Thus the error rate expression becomes

$$P_E(M) = \int_0^\infty p(t)\,P_E(t)\,dt \qquad \text{(B-24)}$$

which becomes, upon substituting Eq. (B-21) and (B-23) into (B-24),

$$P_E(M) \sim \frac{1}{\sqrt{2\pi d}} \int_0^\infty \left[\frac{t}{a}\right]^{M-1} \exp\left[-\frac{(1+d)t^2+a^2}{2}\right] \times I_{M-1}(at)\,dt \qquad \text{(B-25)}$$

Integration of Eq. (B-25) may be performed by using the well-known result (Ref. 26)

$$\int_0^\infty \exp\left[-a^2x^2\right] x^{\mu-1} I_v(bx)dx = \frac{b^v\Gamma\left(\frac{\mu+v}{2}\right)}{2^{v+1}a^{\mu+v}\,\Gamma(v+1)} F\left(\frac{\mu+v}{2}, v+1, \frac{b^2}{4a^2}\right) \qquad \text{(B-26)}$$

where $F(a,c,z)$ is the hypergeometric function defined by Eq. (24). Substituting this into Eq. (B-21) gives

$$P_E(M) \sim \frac{1}{2\sqrt{\pi d}}\left(\frac{1}{1+d}\right)^M \sqrt{1+d}\,\exp\left[-L\right] \times \left[\frac{\Gamma\frac{(2M-1)}{2}}{\Gamma(M)} F\left(\frac{2M-1}{2}, M, \frac{L}{1+d}\right)\right] \qquad \text{(B-27)}$$

If we now use the asymptotic expansion for the hypergeometric function $F(a,c,z)$, namely,

$$F(a,c,z) \sim \frac{\Gamma(c)}{\Gamma(a)} \frac{\exp(z)}{z^{c-a}}\left[1 + 0\left(\frac{1}{z}\right)\right] \qquad \text{(B-28)}$$

Eq. (B-27) reduces to

$$P_E(M) \sim \frac{\exp\left[-\frac{Ld}{1+d}\right]}{\sqrt{4\pi Ld}}\left[\frac{1}{1+d}\right]^{M-1} \qquad \text{(B-29)}$$

which becomes Eq. (65) when the definitions of d, L, β and ρ are used.

APPENDIX C

Eq. (53) may be written as

$$P_E(M) = \sum_{m=0}^{M-1} \frac{1}{m!(1+\beta)} \left(\frac{1}{L\beta}\right)^{\frac{M-1}{2}} \left[-\frac{L\beta}{1+\beta}\right]$$

$$\times \int_0^\infty Y^{M+2m} \exp\left[-\left(\frac{2+\beta}{1+\beta}\right)Y^2\right]$$

$$\times I_{M-1}\left[\sqrt{\frac{4L\beta Y^2}{(1+\beta)^2}}\right] dY \qquad \text{(C-1)}$$

Using Eq. (B-1) to integrate Eq. (C-1) we obtain, after some algebra,

$$P_E(M) = \left[\frac{1}{2+\beta}\right]^M \exp\left[-\frac{L\beta}{2+\beta}\right] \sum_{m=0}^{M-1} \binom{M+m-1}{m}$$

$$\times \left[\frac{1+\beta}{2+\beta}\right]^m F\left(-m, M \frac{-L\beta}{(1+\beta)(2+\beta)}\right) \qquad \text{(C-2)}$$

which is Eq. (54). For large random components or small specular components, the hypergeometric function may be approximated by unity, i.e., $\beta >> 1$. Under such multichannel conditions we write

$$P_E(M) \sim \left[\frac{1}{2+\beta}\right]^M \exp\,[-L] \sum_{m=0}^{M-1} \binom{M+m-1}{m} \left(\frac{1+\beta}{2+\beta}\right)^m \qquad \text{(C-3)}$$

where

$$L = \sum_{i=1}^{M} \gamma_i^2$$

Using a technique developed by Pierce (Ref. 6), we write

$$\frac{1+\beta}{2+\beta} = 1 - \frac{1}{1+\beta}$$

and make a binomial expansion of each term in the sum. Such a procedure yields, after rearrangement of the terms,

$$P_E(M) = \left[\frac{1}{2+\beta}\right]^M \exp\,[-L]$$

$$\times \sum_{m=0}^{M-1} \frac{(-1)^m (2M-1)!}{(M-1)!\,(M-m-1)!\,m!(M+m)} \left[\frac{1}{2+\beta}\right]^M \qquad \text{(C-4)}$$

At large signal-to-noise ratios (small error probability) the first term in the sum predominates and we have to a first approximation

$$P_E(M) \sim \frac{1}{2}\binom{2M}{M} \exp\,[-L] \left[\frac{1}{\beta}\right]^M \qquad \text{(C-5)}$$

Substituting for L gives Eq. (68). If we let $\gamma_i^2 = 0$ for all $i = 1, 2, \cdots M$ we get Eq. (67).

Finally, we wish to derive Eq. (70). For this case we assume that the multichannel possesses large specular components such that $\rho_i > \beta$. Using the first term of the asymptotic expansion for $F(a,c,z)$ [see Eq. (B-28)] and Eq. (C-2) we obtain

$$P_E(M) < \left[\frac{1}{2+\beta}\right]^M \exp\left[\frac{-L\beta}{2+\beta}\right] \sum_{m=0}^{M-1} \frac{1}{m!} \left[\left(\frac{1+\beta}{2+\beta}\right) x\right]^m \qquad \text{(C-6)}$$

where

$$x = \frac{L\beta}{(1+\beta)\,(2+\beta)}$$

Now

$$1 \leq \sum_{m=0}^{M-1} \frac{1}{m!} \left[\left(\frac{1+\beta}{2+\beta}\right) x\right]^m < \exp\left[\left(\frac{1+\beta}{2+\beta}\right) x\right] \qquad \text{(C-7)}$$

and we have the following upper bound:

$$P_E(M) < \left[\frac{1}{2+\beta}\right]^M \exp\left[\frac{-L\beta}{2+\beta}\left(1 - \frac{1}{2+\beta}\right)\right] \qquad \text{(C-8)}$$

Substituting for L gives the required result. If we let $\beta = 0$, substitute for L, and rewrite, we get Eq. (70).

REFERENCES

1. Price, Robert, "Error Probabilities for Adaptive Multichannel Reception of Binary Signals," *IRE Transactions on Information Theory*, Vol. IT-8, No. 5, September 1962.
2. "Scatter Propagation Issue," *Proceedings of the IRE*, October 1955.
3. "Ionospheric Scatter Transmission" and "Tropospheric Scatter Transmission," *Proceedings of the IRE*, January 1960, pp. 4-44.
4. Turin, G.L., "Communication Through Noisy Random-Multipath Channels," *1956 IRE Convention Record*, Part 4, Automatic Control, Circuit Theory and Information Theory, pp. 154-166.
5. Turin, G. L., "Some Computations of Error Rates for Multipath Channels," in *Proceedings of the National Electronics Conference*, Chicago, Ill., October 12-14, 1959, Vol. XV, pp. 431-440, National Electronics Conference, Inc., Chicago, Ill.
6. Pierce, John N., "Theoretical Diversity Improvement in Frequency Shift Keying," *Proceedings of the IRE*, May 1958, pp. 903-910.
7. Barrow, Bruce B., *Error Probabilities for Data Transmission Over Fading Radio Paths*, TM-26, Shape Air Defense Technical Center, Dee Haag, Nederland, February 1962.
8. Pierce, J. N., and S. Stein, "Multiple Diversity with Nonindependent Fading," *Proceedings of the IRE*, Vol. 48, January 1960, pp. 89-104.
9. Turin, G. L., "Error Probabilities for Binary Symmetric Ideal Reception Through Nonselective Slow Fading and Noise," *Proceedings of the IRE*, September 1958, pp. 1603-1619.
10. Rice, S. O., "Mathematical Analysis of Random Noise," *Bell System Technical Journal*, Vol. 24, 1945.
11. Shoemaker, E. M., "Exploration of the Moon's Surface," *American Scientist*, March 1962.
12. Senior, T. B. A., and K. M. Siegel, "A Theory of Radar Scattering by the Moon," *National Bureau of Standards, Journal of Research of the*, Vol. 64D, No. 3, 1960, pp. 217-229.
13. Kailath, T., *Communication via Randomly Varying Channels*, Sc. D. Thesis, Massachusetts Institute of Technology, Dept. of Electrical Engineering, Cambridge, June 1962.
14. Brennan, D. G., "Linear Diversity Combining Techniques," *Proceedings of the IRE*, Vol. 47, June 1959, pp. 1075-1102.
15. Lindsey, W. C., *A Wideband Adaptive Communication System*, Ph. D. Thesis, Purdue University, School of Electrical Engineering, Lafayette, Ind., June 1962.
16. Marcum, J. I., *Table of Q Functions, Report* RM-339, Rand Corporation, Santa Monica, Calif., January 1, 1950.
17. Helstrom, C. W., *Statistical Theory of Signal Detection*, Pergamon Press, Inc., New York, 1960.
18. Montgomery, G. F., "A Comparison of Amplitude and Angle Modulation for Narrow-Band Communication of Binary Coded Messages in Fluctuation Noise," *Proceedings of the IRE*, February 1954.
19. Hahn, P. M., "Theoretical Diversity Improvement in Multiple Frequency Shift Keying," *IRE Transactions on Communication Systems*, Vol. CS-10, No. 2, June 1962.
20. Reiger, S., "Error Rates in Data Transmission," *Proceedings of the IRE*, Vol. 46, May 1958, pp. 919-920.
21. Reiger, S., "Error Probability of Binary Data Transmission in the Presence of Random Noise," *1953 IRE Convention Record*, Part 8, p. 72.
22. Turin, G., "The Asymptotic Behavior of Ideal M-ary Systems," *Proceedings of the IRE*, Vol. 27, No. 1, January 1959.
23. Erdelyi, A., W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill Book Company, Inc., New York.
24. Lindsey, W. C., *Asymptotic Performance Characteristics for Multireceiver Communications Through the Rician Multichannel*, Technical Report No. 32-440, Jet Propulsion Laboratory, Pasadena, Calif., April 30, 1963. (Also to be published in the *IEEE Journal on Communications and Electronics*.)
25. Campbell, G., and R. Foster, *Fourier Integrals*, D. Van Nostrand Co., Inc., Princeton, N. J., 1942.
26. Watson, G. N., *Theory of Bessel Functions*, Cambridge University Press, New York, 1958.